Hamiltonian Analysis of Lagrange Multiplier Modified Gravity

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Abstract

We develop Hamiltonian formalism for Lagrange Multiplier Modified Gravity. We further calculate the Poisson brackets between constraints and we show that they coincide with the algebra of constraints in Hamiltonian formulation of General Relativity.

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1 Introduction

One of the most important problem in present cosmology is the understanding of the origin of the late-time cosmic acceleration (the so-called Dark Energy (DE) epoch). Recently new interesting model DE model was proposed in [1, 2]. This model consists of two scalar fields where one of scalars represents the Lagrange multiplier. The multiplier puts constraint on the second scalar field and as a result the theory contains singe degrees of freedom. It was shown that the energy of the system flows along time-like geodetic that is similar to the dust, however the theory contains non-zero energy. The behavior of this system suggests that it can be natural candidate for unification of Dark Energy and Dark Matter. The cosmological implications of these models were then analyzed in [4, 5, 6]. The role of Lagrange multipliers in the context of $f(R)$ gravities was studied in [3]. Moreover, the Lagrange multipliers in the context of modified gravity may improve the ultraviolet properties of the covariant Hořava-Lifshitz gravity [9] leading to its renormalizability conjecture [7, 8].

As was shown in all these papers the presence of the Lagrange multipliers in the action has strong impact on the form of the resulting equations of motions. Then it is natural to ask the question how the presence of Lagrange multipliers modifies Hamiltonian structure of given theory. Moreover, we would like to see whether the Hamiltonian of these systems is again given as a linear combination of constraints and whether these constraints are the first class and their Poisson algebra respects the basic principles of geometrodynamics [10, 11, 12]. It turns out that Hamiltonian structure of given theory is very interesting. We show that the presence of the first scalar field that plays the role of the Lagrange multiplier implies an existence of the second class constraints. Then after their solving we find the Hamiltonian equations of motions for the second scalar field that are autonomous in the sense that the time evolution of the scalar field does not depend on its conjugate momenta. Such systems were studied in the past especially in the context of the 't Hooft deterministic approach to quantum mechanics [13, 14, 15, 16]. We also find that the resulting theory is a fully constrained system with the algebra of constraints that has the same form as in General Relativity.

As the second example of the Lagrange multiplier modified theory we consider the gravity action introduced in [3]. This action is the Lagrange modification of $F(R)$ gravity theories [2]. We show that the resulting Hamiltonian is given as a linear combination of constraints and has similar structure as the Hamiltonian of $F(R)$ gravities [21, 22] [3]. However there is an important difference that follows the fact that the presence of the Lagrange multiplier implies that the original auxiliary fields become dynamical in Hamiltonian formulation. We further determine the Poisson brackets between constraints. We show that the algebra of these constraints takes exactly the same form as in [10, 11, 12]. In other words we explicitly prove the consistency of Lagrange modified theories of gravity from the Hamiltonian point of view.

[2]For review, see [17, 19, 20].

[3]For related works, see [24, 25].

Let us summarize our results. We study the Lagrange multiplier modified theories with emphasis on their Hamiltonian formalism. We find that the resulting Hamiltonian is again given as a linear combination of the first class constraints. We show that the Poisson brackets of these constraints have the same form as in General Relativity.

This paper is organized as follows. In the next section (2) we perform the Hamiltonian formulation of the General Relativity action together with the Lagrange multiplier modified scalar field action. We find corresponding Hamiltonian and diffeomorphism constraints and calculate their algebra. In section (3) we study the Lagrange multiplier modified action introduced in [3]. We again determine corresponding Hamiltonian. Then we calculate the Poisson brackets of the secondary constraints and we find that they take exactly the same form as in General Relativity.

2 Lagrange Multiplier Modified Scalar Field Action

In this section we develop the Hamiltonian formalism for Lagrange multiplier modified scalar field action. We study the form of the action that was introduced in [3]

$$S=\int d^{(D+1)}x\sqrt{-\hat{g}}[{}^{(D+1)}R(\hat{g})-\frac{\omega(\phi)}{2}\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V(\phi)-\lambda[\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+U(\phi)]] \ . \tag{1}$$

Let us explain our notation. We consider $D+1$ dimensional manifold $\mathcal{M}$ with the coordinates x^μ , $\mu=0,\dots,D$ and where $x^\mu=(t,\mathbf{x})$, $\mathbf{x}=(x^1,\dots,x^D)$. We presume that this space-time is endowed with the metric $\hat{g}_{\mu\nu}(x^\rho)$ with signature $(-,+,\dots,+)$. Suppose that $\mathcal{M}$ can be foliated by a family of space-like surfaces Σ_t defined by $t=x^0$. Let $g_{ij}, i,j=1,\dots,D$ denotes the metric on Σ_t with inverse g^{ij} so that $g_{ij}g^{jk}=\delta_i^k$. We introduce the future-pointing unit normal vector n^μ to the surface Σ_t. In ADM variables we have $n^0=\sqrt{-\hat{g}^{00}}, n^i=-\hat{g}^{0i}/\sqrt{-\hat{g}^{00}}$. We also define the lapse function $N=1/\sqrt{-\hat{g}^{00}}$ and the shift function $N^i=-\hat{g}^{0i}/\hat{g}^{00}$. In terms of these variables we write the components of the metric $\hat{g}_{\mu\nu}$ as

$$\begin{aligned}
&\hat{g}_{00}=-N^2+N_ig^{ij}N_j \ , \quad \hat{g}_{0i}=N_i \ , \quad \hat{g}_{ij}=g_{ij} \ , \\
&\hat{g}^{00}=-\frac{1}{N^2} \ , \quad \hat{g}^{0i}=\frac{N^i}{N^2} \ , \quad \hat{g}^{ij}=g^{ij}-\frac{N^iN^j}{N^2} \ .
\end{aligned} \tag{2}$$

Then it is easy to see that

$$\sqrt{-\det\hat{g}}=N\sqrt{\det g} \ , \quad \hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi=-(\nabla_n\phi)^2+g^{ij}\partial_i\phi\partial_j\phi \ . \tag{3}$$

Further, $D+1$-dimensional curvature ${}^{(D+1)}R$ can be written as

$${}^{(D+1)}R=K^{ij}K_{ij}-K^2+R^{(D)}+\frac{2}{\sqrt{-\hat{g}}}\partial_\mu(\sqrt{-\hat{g}}n^\mu K)-\frac{2}{\sqrt{g}N}\partial_i(\sqrt{g}g^{ij}\partial_jN) \ , \tag{4}$$

where $K_{ij} = \frac{1}{2N}(\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i)$ and where ∇_i is covariant derivative defined using the metric g_{ij}. Further, $K = g^{ij} K_{ji}$. In what follows we ignore these boundary terms when we will presume appropriate boundary conditions. Let us consider the scalar field action. Using the notation introduced above we find the momentum conjugate to ϕ and λ

$$p_\phi = \sqrt{g}(\omega + \lambda)\nabla_n \phi \ , \quad p_\lambda \approx 0 \ . \tag{5}$$

Then the Hamiltonian for the scalar field takes the form

$$\begin{aligned} H^\phi &= \int d^D \mathbf{x} \mathcal{H}^\phi \ , \quad \mathcal{H}^\phi = N \mathcal{H}_T^\phi + N^i \mathcal{H}_i^\phi \ , \quad \mathcal{H}_i = p_\phi \partial_i \phi \ , \\ \mathcal{H}_T^\phi &= \frac{1}{2\sqrt{g}(\omega+\lambda)} p_\phi^2 + \frac{1}{2}\sqrt{g}(\omega+\lambda) g^{ij}\partial_i\phi\partial_j\phi + \sqrt{g}V + \sqrt{g}\lambda U \ . \end{aligned} \tag{6}$$

Finally we write the Hamiltonian for General Relativity part of the action

$$H^{GR} = \int d^D \mathbf{x} \mathcal{H}^{GR} \ , \quad \mathcal{H}^{GR} = N \mathcal{H}_T^{GR} + N^i \mathcal{H}_i^{GR} \ , \tag{7}$$

where

$$\begin{aligned} \mathcal{H}_T^{GR} &= \frac{1}{\sqrt{g}} \pi^{ij} g_{ik} g_{jl} \pi^{kl} - \frac{1}{\sqrt{g} D} \pi^2 - \sqrt{g} R^{(D)} \ , \\ \mathcal{H}_i^{GR} &= -2 g_{ik} \nabla_j \pi^{kj} \ , \end{aligned} \tag{8}$$

where π^{ij} is momentum conjugate to g_{ij} with non-trivial Poisson brackets

$$\left\{ g_{ij}(\mathbf{x}), \pi^{kl}(\mathbf{y}) \right\} = \frac{1}{2}(\delta_i^k \delta_j^l + \delta_i^l \delta_j^k)\delta(\mathbf{x} - \mathbf{y}) \ , \tag{9}$$

and where $\pi \equiv \pi^{ij} g_{ji}$. Note that ∇_i is a covariant derivative calculated with the metric g_{ij} that also obeys $\nabla_i g_{jk} = 0$.

In summary, the total Hamiltonian is $H = H^\phi + H^{GR}$. The preservation of the primary constraints $p_N \approx 0$, $p^i \approx 0$ implies the secondary ones

$$\mathcal{H}_T = \mathcal{H}_T^{GR} + \mathcal{H}_T^\phi \approx 0 \ , \quad \mathcal{H}_i = \mathcal{H}_i^{GR} + \mathcal{H}_i^\phi \approx 0 \ . \tag{10}$$

It is useful to introduce the smeared form of these constraints

$$\begin{aligned} \mathbf{T}_T(N) &= \mathbf{T}_T^{GR}(N) + \mathbf{T}_T^\phi(N) \ , \\ \mathbf{T}_S(N^i) &= \mathbf{T}_S^{GR}(N^i) + \mathbf{T}_S^\phi(N^i) \ , \end{aligned} \tag{11}$$

where

$$\begin{aligned}
\mathbf{T}_T^{GR}(N) &= \int d^D\mathbf{x} N\mathcal{H}_T^{GR} \ , \quad \mathbf{T}_T^{\phi}(N) = \int d^D\mathbf{x} N\mathcal{H}_T^{\phi} \ , \\
\mathbf{T}_S^{GR}(N^i) &= \int d^D\mathbf{x} N^i\mathcal{H}_i^{GR} \ , \quad \mathbf{T}_S^{\phi}(N^i) = \int d^D\mathbf{x}(N^i\mathcal{H}_i^{\phi} + N^i p_\lambda \partial_i\lambda) \ ,
\end{aligned} \tag{12}$$

where we included the primary constraint $p_\lambda \approx 0$ into definition of $\mathbf{T}_S^{\phi}(N^i)$ in order to ensure the correct form of the Poisson bracket between the diffeomorphism generator $\mathbf{T}_S^{\phi}(N^i)$ and the scalar field λ.

It is well known that the Poisson brackets between smeared form of the General Relativity constraints take the form [10, 11, 12]

$$\begin{aligned}
\left\{\mathbf{T}_T^{GR}(N), \mathbf{T}_T^{GR}(M)\right\} &= \mathbf{T}_S^{GR}(g^{ij}(N\partial_j M - M\partial_j N)) \ , \\
\left\{\mathbf{T}_S^{GR}(N^i), \mathbf{T}_T^{GR}(M)\right\} &= \mathbf{T}_T^{GR}(N^i\partial_i M) \ , \\
\left\{\mathbf{T}_S^{GR}(N^i), \mathbf{T}_S^{GR}(M^i)\right\} &= \mathbf{T}_S^{GR}(N^j\partial_j M^i - M^j\partial_j N^i) \ .
\end{aligned} \tag{13}$$

On the other hand we have to determine the Poisson brackets between constraints corresponding to the scalar field. First of all it is easy to see that

$$\left\{\mathbf{T}_S^{\phi}(N^i), \mathbf{T}_S^{\phi}(M^i)\right\} = \mathbf{T}_S^{\phi}(N^j\partial_j M^i - M^j\partial_j N^i) \ . \tag{14}$$

On the other hand the Poisson bracket between $\mathbf{T}_S(N^i)$ and $\mathbf{T}_T^{\phi}(M)$ is equal to

$$\begin{aligned}
\left\{\mathbf{T}_S(N^i), \mathbf{T}_T^{\phi}(M)\right\} &= \int d^D\mathbf{x}(-N^k\partial_k\mathcal{H}_T^{\phi} - \partial_k N^k\mathcal{H}_T^{\phi}) = \\
&= \int d^D\mathbf{x} N^k\partial_k\mathcal{H}_T^{\phi} = \mathbf{T}_T^{\phi}(N^k\partial_k M)
\end{aligned} \tag{15}$$

using

$$\begin{aligned}
\left\{\mathbf{T}_S(N^i), g_{ij}\right\} &= -N^k\partial_k g_{ij} - \partial_i\xi^k g_{kj} - g_{ik}\partial_j\xi^k \ , \\
\left\{\mathbf{T}_S(N^i), \sqrt{g}\right\} &= -N^k\partial_k\sqrt{g} - \sqrt{g}\partial_k N^k \ .
\end{aligned} \tag{16}$$

Note that the presence of the term $N^i p_\lambda \partial_i\lambda$ in the definition of $\mathbf{T}_S^{\phi}(N^i)$ was crucial for deriving of the correct form of the Poisson bracket (15). Finally we calculate the Poisson bracket between $\mathbf{T}_T^{\phi}(N), \mathbf{T}_T^{\phi}(M)$ and after some algebra we find the desired result

$$\left\{\mathbf{T}_T^{\phi}(N), \mathbf{T}_T^{\phi}(M)\right\} = \mathbf{T}_S^{\phi}(g^{ij}(N\partial_j M - M\partial_j N)) \ . \tag{17}$$

It is also easy to show that

$$\left\{\mathbf{T}_T^{GR}(N), \mathbf{T}_T^{\phi}(M)\right\} + \left\{\mathbf{T}_T^{\phi}(N), \mathbf{T}_T^{GR}(M)\right\} = 0 \tag{18}$$

due to the fact that $\mathcal{H}_T^\phi$ depends on g and not on their derivatives. If we combine these results we find that the Poisson brackets of the constraints $\mathbf{T}_T(N), \mathbf{T}_S(N^i)$ has the desired form (13).

As the next step we analyze the stability of the primary constraint $p_\lambda \approx 0$. The requirement of its stability implies the secondary constraint

$$\partial_t p_\lambda(\mathbf{x}) = \{p_\lambda(\mathbf{x}), H\} = \frac{1}{2\sqrt{g}(\omega+\lambda)^2} p_\phi^2 - \frac{1}{2}\sqrt{g} g^{ij}\partial_i\phi\partial_j\phi - \sqrt{g}U \equiv \mathcal{G}_\lambda(\mathbf{x}) \approx 0 \ . \tag{19}$$

We observe that

$$\{p_\lambda(\mathbf{x}), \mathcal{G}_\lambda(\mathbf{y})\} = \frac{1}{\sqrt{g}(\omega+\lambda)^3} p_\phi^2(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) \ . \tag{20}$$

In other words p_λ and $\mathcal{G}_\lambda$ are the second class constraints. However there are additional non-zero Poisson brackets. The first one is

$$\begin{aligned} \{\mathcal{G}_\lambda(\mathbf{x}), \mathcal{G}_\lambda(\mathbf{y})\} &= -2\frac{1}{\sqrt{g}(\omega+\lambda)^2} g^{ij}\mathcal{H}_j^\phi(\mathbf{x})\partial_i\delta(\mathbf{x}-\mathbf{y}) - \\ &- \partial_i\left[\frac{1}{\sqrt{g}(\omega+\lambda)^2} g^{ij}\mathcal{H}_j^\phi(\mathbf{x})\right]\delta(\mathbf{x}-\mathbf{y}) \ . \end{aligned} \tag{21}$$

It is also clear from the structure of the constraint $\mathcal{G}_\lambda$ that there is non-zero Poisson brackets between $\mathcal{G}_\lambda$ and $\mathcal{H}$ defined by (10)

$$\{\mathcal{G}_\lambda(\mathbf{x}), \mathcal{H}\} \neq 0 \ , \tag{22}$$

where $\mathcal{H} = N\mathcal{H}_T + N^i\mathcal{H}_i$. Note that the explicit form of this Poisson bracket is not important for us.

Using these results we can proceed to the study of the stability of the secondary constraints. Following the standard analysis of the constraint systems we introduce the total Hamiltonian as

$$H_T = H + \int d^D\mathbf{x}(\alpha p_\lambda + \beta\mathcal{G}_\lambda) \ , \tag{23}$$

where α, β are Lagrange multipliers and analyze the stability of the constraints $\mathcal{H}, \mathcal{G}_\lambda, p_\lambda$. Firstly we have

$$\begin{aligned} \partial_t\mathcal{H} &= \{\mathcal{H}(\mathbf{x}), H\} + \int d^D\mathbf{y}(\beta\{\mathcal{H}(\mathbf{x}), \mathcal{G}_\lambda(\mathbf{y})\} + \beta\{\mathcal{H}(\mathbf{x}), p_\lambda(\mathbf{y})\}) \approx \\ &\approx \int d^D\mathbf{y}\beta\{\mathcal{H}(\mathbf{x}), \mathcal{G}_\lambda(\mathbf{y})\} \neq 0 \ , \end{aligned} \tag{24}$$

where we used the fact that the Poisson brackets between $\mathcal{H}$ and H weakly vanish. Then the requirement of stability of the constraint $\mathcal{H} \approx 0$ determines the value of

the Lagrange multiplier β to be equal to 0. On the other hand the time evolution of the constraint $\mathcal{G}_\lambda$ is given by the equation

$$\partial_t \mathcal{G}_\lambda(\mathbf{x}) = \{\mathcal{G}_\lambda(\mathbf{x}), H\} + \int d^D\mathbf{y}\alpha(\mathbf{y}) \{\mathcal{G}_\lambda(\mathbf{x}), p_\lambda(\mathbf{y})\} \approx 0 \ . \tag{25}$$

Due to the fact that $\{\mathcal{G}_A, H\} \neq 0$ and $\{\mathcal{G}_\lambda, p_\lambda\} \neq 0$ the equation above can be solved for α at least in principle. Then using these results it is easy to see that the constraint $p_\lambda \approx 0$ is preserved during the time evolution of the system. Further, p_λ and $\mathcal{G}_\lambda$ are the second class constraints that can be solved for λ and p_λ so that the reduced phase space is spanned by $(g_{ij}, \pi^{ij}), (\phi, p_\phi)$ and the symplectic structure is given by the Dirac brackets between these variables. In order to find their form we introduce following notation for the Poisson brackets of the second class constraints $p_\lambda, \mathcal{G}_\lambda$

$$\begin{aligned}
\triangle_{11}(\mathbf{x},\mathbf{y}) &= \{p_\lambda(\mathbf{x}), p_\lambda(\mathbf{y})\} = 0 \ , \quad \triangle_{12}(\mathbf{x},\mathbf{y}) = \{p_\lambda(\mathbf{x}), \mathcal{G}_\lambda(\mathbf{y})\} \neq 0 \ , \\
\triangle_{21}(\mathbf{x},\mathbf{y}) &= \{\mathcal{G}_\lambda(\mathbf{x}), p_\lambda(\mathbf{y})\} \neq 0 \ , \quad \triangle_{22}(\mathbf{x},\mathbf{y}) = \{\mathcal{G}_\lambda(\mathbf{x}), \mathcal{G}_\lambda(\mathbf{y})\} \neq 0
\end{aligned} \tag{26}$$

and denote the inverse matrix as $(\triangle^{-1})^{AB}(\mathbf{x},\mathbf{y})$. This matrix by definition obeys the equation

$$\int d^{\mathbf{x}} \triangle_{AC}(\mathbf{x},\mathbf{z})(\triangle^{-1})^{CB}(\mathbf{z},\mathbf{y}) = \delta_A^B \delta(\mathbf{x}-\mathbf{y}) \ . \tag{27}$$

It can be shown that the matrix $(\triangle^{-1})$ has following structure

$$(\triangle^{-1}) = \left(\begin{array}{cc} * & * \\ * & 0 \end{array} \right) \ , \tag{28}$$

where $*$ denotes non-zero elements. It is important for the calculation of the Dirac brackets that $(\triangle^{-1})^{22} = 0$. Explicitly, the Dirac bracket between ϕ and p_ϕ takes the form

$$\begin{aligned}
\{\phi(\mathbf{x}), p_\phi(\mathbf{y})\}_D &= \{\phi(\mathbf{x}), p_\phi(\mathbf{y})\} - \\
&- \int d^D\mathbf{z} d^D\mathbf{z}' \{\phi(\mathbf{x}), \Phi_A(\mathbf{z})\} (\triangle^{-1})^{AB}(\mathbf{z},\mathbf{z}') \{\Phi_B(\mathbf{z}'), p_\phi(\mathbf{y})\} = \\
&= \{\phi(\mathbf{x}), p_\phi(\mathbf{y})\} - \int d^D\mathbf{z} d^D\mathbf{z}' \{\phi(\mathbf{x}), \mathcal{G}_\lambda(\mathbf{z})\} (\triangle^{-1})^{22}(\mathbf{z},\mathbf{z}') \{\mathcal{G}_\lambda(\mathbf{z}'), p_\phi(\mathbf{y})\} = \\
&= \{\phi(\mathbf{x}), p_\phi(\mathbf{y})\} \ ,
\end{aligned} \tag{29}$$

where $\Phi_A = (p_\lambda, \mathcal{G}_\lambda)$ is the common notation for the second class constraints.

We are now ready to completely eliminate the second class constraints Φ_A. The constraint $\mathcal{G}_\lambda = 0$ can be solved for $\omega + \lambda$

$$(\omega + \lambda) = \frac{p_\phi}{\sqrt{g}\sqrt{g^{ij}\partial_i\phi\partial_j\phi + 2U}} \ . \tag{30}$$

Inserting this result into the Hamiltonian constraint (6) we find that it takes the form

$$\mathcal{H}_T^\phi = p_\phi \sqrt{g^{ij}\partial_i\phi\partial_j\phi + 2U} + \sqrt{g}V - \sqrt{g}\omega U \ . \tag{31}$$

We observe that this Hamiltonian density is linear in momenta. Then the equation of motion for ϕ takes the form

$$\partial_t\phi = \{\phi, H\} = N\sqrt{g^{ij}\partial_i\phi\partial_j\phi + 2U} + N^i\partial_i\phi \tag{32}$$

that shows that the time evolution of ϕ does not depend on p_ϕ. Such systems were extensively studied in the past in the context of 't Hooft's deterministic approach to quantum mechanics [13, 14, 15, 16] and it is really interesting that the Hamiltonian

with similar structure arises in Lagrange modified multiplier theory. [4] We complete our analysis by calculation of the Poisson bracket between $\mathcal{H}_T^\phi$ given in (31) and the spatial diffeomorphism constraint $\mathbf{T}_S(N^i)$. Using

$$\left\{\mathbf{T}_S(N^i), g^{ij}\partial_i\phi\partial_j\phi\right\} = -N^k\partial_k\left(g^{ij}\partial_i\phi\partial_j\phi\right) \tag{41}$$

we easily find

$$\left\{\mathbf{T}_S(N^i), \mathcal{H}_T^\phi\right\} = -N^i\partial_i\mathcal{H}_T^\phi - \partial_i N^i\mathcal{H}_T^\phi \ , \tag{42}$$

where $\mathcal{H}_T^\phi$ was given in (31). The analysis of the remaining Poisson brackets is the same as above with conclusion that the smeared form of the constraints obey the

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[4] We review the basic facts considering such system, following [18]. Let us consider the Hamiltonian system

$$H = p_i f^i(q) + U(q) \ , i = 1, \dots, N \ . \tag{33}$$

From (33) we determine the equations of motion for q^i

$$\partial_t q^i = \left\{q^i, H\right\} = f^i(q) \ . \tag{34}$$

This equation for q^i is autonomous, i.e., it is decoupled from the conjugate momenta p_i. Further it is impossible to perform the Legendre transformation to the Lagrangian since $H_{ij} = \frac{\partial^2 H}{\partial p_i \partial p_j} = 0$. However it is possible to find the Lagrangian that gives the equation of motion (34) when we introduce the auxiliary fields λ_i and write the Lagrangian as

$$L = \lambda_i(\dot{q}^i - f^i(q)) - U(q) \ . \tag{35}$$

No we show that from (35) we can derive the Hamiltonian (33). The momenta conjugate to λ_i and q^i take the form

$$p_\lambda^i = \frac{\delta L}{\delta\dot{\lambda}_i} \approx 0 \ , \quad p_i^q = \frac{\delta L}{\delta\dot{q}^i} = \lambda_i \tag{36}$$

so that we have two sets of primary constraints

$$\Phi_\lambda^i = p_\lambda^i \approx 0 \ , \quad \Phi_i^q = p_i^q - \lambda_i \approx 0 \ . \tag{37}$$

The extended Hamiltonian that follows from (35) takes the form

$$H_E = H + \omega_i^\lambda\Phi_\lambda^i + \omega_q^i\Phi_i^q \ , \quad H = \lambda_i f^i + U(q) \ . \tag{38}$$

Then we study the stability of the constraints Φ_λ^i , Φ_i^q

$$\begin{aligned} \partial_t\Phi_\lambda^i &= \left\{\Phi_\lambda^i, H_E\right\} = -f^i + \omega_q^i = 0 \\ \partial_t\Phi_i^q &= \left\{\Phi_i^q, H_E\right\} = -\lambda_j\frac{df^j}{dq^i} - \omega_i^q = 0 \ . \end{aligned} \tag{39}$$

From these equations we can in principle determine the Lagrange multipliers. In other words the constraints Φ_λ^i, Φ_i^q are the second class that should strongly vanish. The solving of these constraints we find the Hamiltonian

$$H = p_i^q f^i + U(q) \tag{40}$$

that coincides with the Hamiltonian (33). Further, it can be easily shown that the Dirac brackets between q^i and p_i coincide with their Poisson brackets. However the problem with the Hamiltonian (33) is that is not bounded from below which is due to the absence of a leading kinetic term quadratic in the momenta $(p_i)^2$.

algebra of constraints given in (13). In other words we show that the Lagrangian multiplier modified scalar action together with General Relativity action obeys the basis rules of geometrodynamics.

3 Hamilton Analysis of $F(R)$ Theories with Lagrange Multipliers

It turns out that the Lagrange multiplier modified $F(R)$-gravity possesses many interesting properties. For example, the reconstruction programme can be more easily performed in Lagrange multiplier modified gravity [3]. In usual $F(R)$-gravity, we need to solve the complicated differential equation to realize the reconstruction program, for recent review, see [23]. It was demonstrated in [3] that the presence of constraint significantly simplifies the reconstruction scenario. It was also shown there that the presence of Lagrange multiplier implies that it is necessary to include the second $F(R)$ function into action.

The action introduced in [3] takes the form

$$S=\int d^{D+1}x\sqrt{-\hat{g}}\left[F_1({}^{(D+1)}R)-\lambda\left(\frac{1}{2}\partial_\mu{}^{(D+1)}R\hat{g}^{\mu\nu}\partial_\nu{}^{(D+1)}R+F_2({}^{(D+1)}R)\right)\right] . \tag{43}$$

Introducing two auxiliary fields A, B we can rewrite the action (43) into the form

$$S=\int d^{D+1}x\sqrt{-\hat{g}}\left[F_1(A)-\lambda\left(\frac{1}{2}\partial_\mu A\hat{g}^{\mu\nu}\partial_\nu A+F_2(A)\right)+B({}^{(D+1)}R-A)\right] . \tag{44}$$

It is easy to see that integration of A, B from (44) leads to (43). Our goal is to find the Hamiltonian from (44) implementing $D+1$ formalism. In fact using (4) it is easy to see that the action (44) takes the form

$$\begin{aligned} S &= \int d^D\mathbf{x}dt\sqrt{g}N\left(F_1(A)-\lambda(-\nabla_nA\nabla_nA+g^{ij}\partial_iA\partial_jA+F_2(A))-BA\right)+ \\ &+ \int d^D\mathbf{x}dt\sqrt{g}NB(K_{ij}\mathcal{G}^{ijkl}K_{kl}+R^{(D)}-A)- \\ &- -2\int d^D\mathbf{x}dt(\sqrt{g}(\partial_tB-N^i\partial_iB)K+2\sqrt{g}\partial_iBg^{ij}\partial_jN) , \end{aligned} \tag{45}$$

where we performed integration by parts and ignored boundary terms. From (45) we easily find momenta conjugate to canonical variables g_{ij}, N, N_i, A and B

$$\begin{aligned} \pi^{ij} &= \sqrt{g}B\mathcal{G}^{ijkl}K_{kl}-\sqrt{g}\nabla_nBg^{ij} , \quad ,p_N\approx 0 , \quad ,p^i\approx 0 , \\ p_B &= -2\sqrt{g}K , \quad p_A=2\sqrt{\lambda}\nabla_nA , \quad p_\lambda\approx 0 . \end{aligned} \tag{46}$$

Note that the Lagrange multiplier implies that A is a dynamical field which is different from standard $F(R)$ theory of gravity where A remains auxiliary field. Then after some effort we derive the Hamiltonian density in the form

$$\mathcal{H} = N\mathcal{H}_T + N^i\mathcal{H}_i \ , \tag{47}$$

where

$$\begin{aligned}\mathcal{H}_T &= \frac{1}{\sqrt{g}B}\pi^{ij}g_{ik}g_{il}\pi^{kl} - \frac{1}{\sqrt{g}BD}\pi^2 - \frac{\pi p_B}{\sqrt{g}D} + \\ &+ \frac{B}{4\sqrt{g}D}(D-1)p_B^2 - \sqrt{g}BR^{(D)} + 2\partial_i[\sqrt{g}g^{ij}\partial_j B] \\ &+ \frac{1}{4\sqrt{g}\lambda}p_A^2 + \sqrt{g}BA - \sqrt{g}[F_1(A) - \lambda(g^{ij}\partial_i A\partial_j A + F_2(A))] \ ,\end{aligned} \tag{48}$$

and where

$$\mathcal{H}_i = p_A\partial_i A + p_B\partial_i B + p_\lambda\partial_i\lambda - 2g_{ik}\nabla_j\pi^{jk} \ . \tag{49}$$

For further purposes we split $\mathcal{H}_T$ into two parts as $\mathcal{H}_T = \mathcal{H}_T^{GR} + \mathcal{H}_T^A$ where

$$\begin{aligned}\mathcal{H}_T^{GR} &= \frac{1}{\sqrt{g}B}\pi^{ij}g_{ik}g_{il}\pi^{kl} - \frac{1}{\sqrt{g}BD}\pi^2 - \frac{\pi p_B}{\sqrt{g}D} + \\ &+ \frac{B}{4\sqrt{g}D}(D-1)p_B^2 - \sqrt{g}BR^{(D)} + 2\partial_i[\sqrt{g}g^{ij}\partial_j B] \\ \mathcal{H}_T^A &= \frac{1}{4\sqrt{g}\lambda}p_A^2 + \sqrt{g}BA - \sqrt{g}[F_1(A) - \lambda(g^{ij}\partial_i A\partial_j A + F_2(A))] \ .\end{aligned} \tag{50}$$

The theory possesses $2 + D$ primary constraints

$$\pi_N \approx 0 \ , \pi_i \approx 0 \ , \pi_\lambda \approx 0 \ . \tag{51}$$

The preservation of the primary constraints π_N and π_i imply the secondary constraints $\mathcal{H}_T \approx 0$, $\mathcal{H}_i \approx 0$ while the preservation of $\pi_\lambda \approx 0$ leads to the secondary constraint

$$\mathcal{G}_\lambda = \frac{1}{4\sqrt{g}\lambda^2}p_A^2 - \sqrt{g}(g^{ij}\partial_i A\partial_j A + F_2(A)) \approx 0 \tag{52}$$

We see that it takes the same form as the secondary constraint (19). Clearly p_λ together with $\mathcal{G}_\lambda$ are the second class constraints. Properties of these constraints were analyzed in previous section and results derived there can be used in this section as well.

On the other hand the form of the Hamiltonian constraint $\mathcal{H}_T^{GR}$ is new and we have to check that this constraint is preserved during the time evolution of the

system. In other words we have to calculate the Poisson brackets of the smeared form of these constraints [5]

$$\mathbf{T}_T^{GR}(N) = \int d^D\mathbf{x} N(\mathbf{x})\mathcal{H}_T^{GR}(\mathbf{x}) \ , \quad \mathbf{T}_S^{GR}(N^i) = \int d^D\mathbf{x} N^i(\mathbf{x})\mathcal{H}_i^{GR}(\mathbf{x}) \ . \qquad (53)$$

Let us now outline the strategy of the calculations of these Poisson brackets. In the process of their calculations several delta functions occur. However it turns out that the non-zero contributions give terms that contain derivatives of these delta functions. Such expressions arise for example from following Poisson bracket

$$\left\{\pi^{kl}(\mathbf{x}), (\sqrt{g}R^D)(\mathbf{y})\right\} = -\frac{\delta(\sqrt{g}R^{(D)}(\mathbf{y}))}{\delta g_{kl}(\mathbf{x})} \ . \qquad (54)$$

The right side of this equation can be calculated using the formulas

$$\delta R^{(D)} = -(R^{(D)})^{ij}\delta g_{ij} + \nabla^i\nabla^j\delta g_{ij} - g^{ij}\nabla_k\nabla^k\delta g_{ji} \ , \quad \delta g = g g^{ij}\delta g_{ij} \ . \qquad (55)$$

Now we are ready to perform these calculations. It turns out that following non-zero Poisson brackets contribute to the final result

$$\begin{aligned}
& -\left\{\int d^D\mathbf{x} N\frac{1}{B\sqrt{g}}\pi^{ij}g_{ik}g_{jl}\pi^{ij}, \int d^D\mathbf{y} M\sqrt{g}R^{(D)}\right\} - \\
- & \left\{\int d^D\mathbf{y} N\sqrt{g}R^{(D)}, \int d^D\mathbf{x} M\frac{1}{B\sqrt{g}}\pi^{ij}g_{ik}g_{jl}\pi^{ij}\right\} = \\
= & \ 2\int d^D\mathbf{x}(N\nabla_i\nabla_j M - M\nabla_i\nabla_j N)\pi^{ij} - 2\int d^D\mathbf{x}\pi(N\nabla_i\nabla^i M - M\nabla_i\nabla^i N) + \\
+ & \ 4\int d^D\mathbf{x}\pi^{ij}(N\nabla_i M - M\nabla_i N)\pi^{ij}\frac{1}{B}\nabla_j B - 4\int d^D\mathbf{x}\pi(N\nabla_i M - M\nabla_i N)\frac{1}{B}\nabla^i B \ ,
\end{aligned} \qquad (56)$$

$$\begin{aligned}
& \left\{\int d^D\mathbf{x}\frac{N}{\sqrt{g}B}\pi^2, \int d^D\mathbf{y} M\sqrt{g}R^{(D)}\right\} + \left\{\int d^D\mathbf{x} N\sqrt{g}R^{(D)}, \int d^D\mathbf{y}\frac{M}{\sqrt{g}B}\pi^2\right\} = \\
= & -\frac{2}{D}\int d^D\mathbf{x}\pi(N\nabla_i\nabla^i M - M\nabla_i\nabla^i N) + 2\int d^D\mathbf{x}\pi(N\nabla_i\nabla^i M - M\nabla_i\nabla^i N) - \\
- & \ \frac{4}{D}\int d^D\mathbf{x}\pi(N\nabla_i M - M\nabla_i N)\frac{\nabla^i B}{B} + 4\int d^D\mathbf{x}\pi(N\nabla_i M - M\nabla_i N)\frac{\nabla^i B}{B}
\end{aligned} \qquad (57)$$

[5] In [22] similar analysis has been performed in the context of non-projectable version of Hořava-Lifshitz $F(R)$ gravity.

and

$$\begin{aligned}
& \left\{\int d^D\mathbf{x}N\frac{\pi p_B}{D\sqrt{g}},\int d^D\mathbf{y}\sqrt{g}R^{(D)}BM\right\}+\left\{\int d^D\mathbf{y}\sqrt{g}R^{(D)}BN,\int d^D\mathbf{x}M\frac{\pi p_B}{D\sqrt{g}}\right\}= \\
= \ & -\frac{(1-D)}{D}\int d^D\mathbf{x}p_BB(N\nabla_i\nabla^iM-M\nabla_i\nabla^iN)- \\
- \ & \frac{2(1-D)}{D}\int d^D\mathbf{x}p_B(N\nabla_iMM-M\nabla_iN)\nabla^iB \ ,
\end{aligned} \tag{58}$$

$$\begin{aligned}
& \left\{\int d^D\mathbf{x}\frac{N}{B\sqrt{g}}\pi^{ij}g_{ik}g_{jl}\pi^{kl},\int d\mathbf{y}2M\partial_i[\sqrt{g}g^{ij}\partial_jB]\right\}+ \\
+ \ & \left\{\int d\mathbf{y}2N\partial_i[\sqrt{g}g^{ij}\partial_jB],\int d^D\mathbf{x}\frac{M}{B\sqrt{g}}\pi^{ij}g_{ik}g_{jl}\pi^{kl}\right\}= \\
= \ & 2\int d^D\mathbf{x}\pi\frac{1}{B}(N\nabla_iM-M\nabla_iN)g^{ij}\nabla_jB-4\int d^D\mathbf{x}\frac{1}{B}(N\nabla_iM-M\nabla_iN)\pi^{ij}\nabla_jB
\end{aligned} \tag{59}$$

and

$$\begin{aligned}
- \ & \left\{\int d^D\mathbf{x}\frac{N}{\sqrt{g}BD}\pi^2,\int d^D\mathbf{y}M2\partial_m[\sqrt{g}g^{mn}\partial_nB]\right\}- \\
- \ & \left\{\int d^D\mathbf{x}N2\partial_m[\sqrt{g}g^{mn}\partial_nB],\int d^D\mathbf{x}\frac{M}{\sqrt{g}BD}\pi^2\right\}= \\
= \ & \frac{2(2-D)}{D}\int d^D\mathbf{x}\frac{\pi}{B}(N\nabla_iM-M\nabla_iN)\nabla^iB
\end{aligned} \tag{60}$$

and

$$\begin{aligned}
- \ & \left\{\int d^D\mathbf{x}\frac{N}{\sqrt{g}D}\pi p_B,\int d^D\mathbf{y}2M\partial_m[\sqrt{g}g^{mn}\partial_nB]\right\}+ \\
- \ & \left\{\int d^D\mathbf{y}2N\partial_m[\sqrt{g}g^{mn}\partial_nB],\int d^D\mathbf{x}\frac{M}{\sqrt{g}D}\pi p_B\right\}= \\
= \ & \frac{(2-D)}{D}\int d^D\mathbf{x}(N\nabla_mM-M\nabla_mN)p_B\nabla^mB+ \\
+ \ & \frac{2}{D}\int d^D\mathbf{x}\pi(N\nabla_m\nabla^mM-M\nabla_m\nabla^mN) \ ,
\end{aligned} \tag{61}$$

$$
\begin{aligned}
& \left\{\int d^D\mathbf{x}\frac{NB}{4\sqrt{g}D}(D-1)p_B^2, 2\int d^D\mathbf{y}M\partial_m[\sqrt{g}g^{mn}\partial_n B]\right\}+ \\
+ & \left\{2\int d^D\mathbf{x}N\partial_m[\sqrt{g}g^{mn}\partial_n B], \int d^D\mathbf{y}\frac{MB}{4\sqrt{g}D}(D-1)p_B^2\right\}= \\
= & -\frac{D-1}{D}\int d^D\mathbf{x}p_B B(N\nabla_m\nabla^m M - M\nabla_m\nabla^m N) \ .
\end{aligned}
\tag{62}
$$

Collecting all these terms together we obtain that almost all contributions cancel and the final result takes the form

$$
\left\{\mathbf{T}_T^{GR}(M), \mathbf{T}_T^{GR}(N)\right\} = \mathbf{T}_S^{GR}((N\nabla_j M - M\nabla_j N)g^{ji}) \ . \tag{63}
$$

In other words the Poisson bracket of the smeared form of the Hamiltonian constraints (50) has the same form as in General Relativity and hence it is with agreement with basic principles of geometrodynamics. Alternatively, it has the form that is expected for fully diffeomorphism invariant theory. Note also that the Poisson bracket between smeared form of the diffeomorphism and Hamiltonian constraint takes the standard form that follows from the fact that Hamiltonian is manifestly invariant under spatial diffeomorphism. Then it is clear that the diffeomorphism and Hamiltonian constraints are preserved during the time evolution of the system.

Now it is straightforward to finish the analysis of the Poisson brackets of the constraints of the Lagrange multiplier modified gravity. Since the Poisson brackets of the constrains corresponding to the gravity part of the action are the same as in General Relativity and since the scalar part of the constraints has exactly the same form as in previous section we immediately find that the Poisson brackets of the Lagrange multiplier modified $F(R)$ gravity take the form

$$
\begin{aligned}
\{\mathbf{T}_T(N), \mathbf{T}_T(M)\} &= \mathbf{T}_S(g^{ij}(N\partial_j M - M\partial_j N)) \ , \\
\left\{\mathbf{T}_S(N^i), \mathbf{T}_T(M)\right\} &= \mathbf{T}_T(N^i\partial_i M) \ , \\
\left\{\mathbf{T}_S(N^i), \mathbf{T}_S(M^i)\right\} &= \mathbf{T}_S(N^j\partial_j M^i - M^j\partial_j N^i) \ .
\end{aligned}
\tag{64}
$$

where

$$
\mathcal{H}_T = \mathcal{H}_T^{GR} + \mathcal{H}_T^A \ , \quad \mathcal{H}_i = -g_{il}\nabla_k\pi^{lk} + p_A\partial_i A \ , \tag{65}
$$

where $\mathcal{H}_T^{GR}$ is given in (48). Note that $\mathcal{H}_T^A$ is equal to

$$
\mathcal{H}_T^A = p_A\sqrt{g^{ij}\partial_i A\partial_j A + F_2(A)} + \sqrt{g}BA - \sqrt{g}F_1(A) \tag{66}
$$

after solving the second class constraint $\mathcal{G}_\lambda$ given in (52) with respect to λ

$$
\lambda = \frac{p_A}{2\sqrt{g}\sqrt{F_2(A) + g^{ij}\partial_i\phi\partial_j\phi}} \ . \tag{67}
$$

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